ISRAEL JOURNAL OF MATHEMATICS 88 (1994), 253–277

ON THE CONJUGACY OF ELEMENT-CONJUGATE HOMOMORPHISMS

BY

MICHAEL LARSEN*

Department of Mathematics, University of Pennsylvania Philadelphia, PA 19104, USA

ABSTRACT

A group G is **acceptable** if a homomorphism ϕ from a finite group Γ to G is determined up to conjugation by the conjugacy classes of the elements $\phi(\gamma)$. Some progress is made toward classifying acceptable Lie groups.

Introduction

Let G denote the set of real points of a linear algebraic group and Γ a finite group. Let $G^\natural$ denote the set of conjugacy classes of G. For any homomorphism $\phi\colon \Gamma \to G$, let $\phi^\natural$ denote the composition of ϕ with the canonical map $G \to G^\natural$. If, for two homomorphisms $\phi_1, \phi_2\colon \Gamma \to G$, $\phi_1^\natural = \phi_2^\natural$, we say that ϕ_1 and ϕ_2 are **element G-conjugate**. We would like to know whether two element-conjugate homomorphisms ϕ_1, ϕ_2 must be **globally G-conjugate**, *i.e.*, whether $\phi^\natural$ determines the isomorphism class of ϕ. We call a Lie group G **acceptable** if element-conjugacy implies global conjugacy for every finite Γ and every pair of homomorphisms $\Gamma \to G$.

The general question of the relation between element-conjugacy and global conjugacy arises in many contexts in algebra, number theory, and geometry. The particular slice of the question considered in this paper was motivated by multiplicity-one questions in the theory of automorphic forms [1]. Thanks to the theorem of Sunada [9], it is closely related to the question of when a compact

* Supported by N.S.F. Grant No. DMS-8807203 and N.S.A. Grant No. MDA 904-92-H-3026.
Received January 28, 1993 and in revised form September 1, 1993

group can be the common covering space of a pair of non-isometric isospectral manifolds. In fact, if ϕ_1, ϕ_2 are element-conjugate homomorphisms $\Gamma \to G$, and the *image groups* $\phi_i(\Gamma)$ are not conjugate in G, then $G/\phi_1(\Gamma)$ and $G/\phi_2(\Gamma)$ are such a pair. Note that if Γ has outer automorphisms, the $\phi_i(\Gamma)$ may be globally conjugate even if the ϕ_i are not. In a subsequent paper, we will give examples of isospectral manifolds arising in this way.

This paper is a step toward the classification of acceptable groups. The first section is devoted to generalities, including the reduction to the case of semisimple groups G. The remaining two sections give an incomplete treatment of the case that G is simple, §2 being devoted to acceptable and §3 to unacceptable groups. All algebraic groups are assumed to be connected unless otherwise specified, though the associated real Lie groups may not be.

As a specimen of the results presented below, the following table summarizes what is now known about the acceptability of the *complex* simple Lie groups:

Root System	Group	Acceptable	Unacceptable	Unknown
A_{n-1}	$\mathrm{SL}(n, \mathbf{C})$	all		
A_{2n-1}	$\mathrm{SL}(2n, \mathbf{C})/\{\pm 1\}$	$n = 1$	$8\|n$	otherwise
A_{mn-1}	$\mathrm{SL}(mn, \mathbf{C})/\mu_m$		all $(m \geq 3)$	
B_n $(n \geq 2)$	$\mathrm{Spin}(2n+1, \mathbf{C})$	$n = 2$	$n \geq 4$	$n = 3$
	$\mathrm{SO}(2n+1, \mathbf{C})$	all		
C_n $(n \geq 3)$	$\mathrm{Sp}(2n, \mathbf{C})$	all		
	$\mathrm{PSp}(2n, \mathbf{C})$		$8\|n$	otherwise
D_n $(n \geq 4)$	$\mathrm{Spin}(2n, \mathbf{C})$		$n \geq 5$	$n = 4$
	$\mathrm{SO}(2n, \mathbf{C})$		all	
	$\mathrm{PSO}(2n, \mathbf{C})$		$8\|n$	otherwise
	Other $(2\|n,\ n \geq 6)$			all
E_6	simply connected			$\surd$
	adjoint			$\surd$
E_7	simply connected			$\surd$
	adjoint			$\surd$
E_8	$E_8(\mathbf{C})$			$\surd$
F_4	$F_4(\mathbf{C})$		$\surd$	
G_2	$G_2(\mathbf{C})$	$\surd$		

ACKNOWLEDGEMENT: I want to thank J. Arthur and D. Blasius for suggesting this problem. I also want to acknowledge useful conversations with E. Calabi, D. DeTurck, and S. Shatz and to thank the referee for his careful reading of and many improvements to an earlier version of this paper.

1. Generalities

Unless otherwise specified, G, G_i, H etc. denote the groups of $\mathbf{R}$-points of connected linear algebraic groups. Of particular interest (see the table above) is the case of groups obtained by Weil restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$.

PROPOSITION 1.1: *Let G_1 and G_2 be two real linear algebraic groups. Then $G = G_1 \times G_2$ is acceptable if and only if G_1 and G_2 are both acceptable.*

Proof: Suppose G_1 and G_2 are acceptable. Let ϕ, ψ: $\Gamma \to G$ be element-conjugate homomorphisms. Let ϕ_i (resp. ψ_i) denote the composition of ϕ (resp. ψ) with the projection map from $G \to G_i$. As $\phi^\natural = \psi^\natural$, it follows that $\phi_i^\natural = \psi_i^\natural$ for $i = 1, 2$. Therefore,

$$\psi_i = g_i \phi_i g_i^{-1}$$

for some $g_i \in G_i$, so

$$\psi = (g_1, g_2)\phi(g_1, g_2)^{-1}.$$

Conversely, if G is acceptable, for $n \in \{1, 2\}$, let ϕ_n, ψ_n: $\Gamma \to G_n$ denote element-conjugate homomorphisms. Let ϕ (resp. ψ) denote the composition of ϕ_n (resp. ψ_n) with the canonical inclusion $G_n \subset G$. Then ϕ and ψ are globally conjugate, so

$$\psi = (g_1, g_2)\phi(g_1, g_2)^{-1}; \quad \psi_n = g_n \phi_n g_n^{-1}.$$

Thus G_n is acceptable. ∎

1.2 Every linear algebraic group G over a field of characteristic zero has a Levi decomposition $G = N \rtimes M$, where M is reductive and N is the unipotent radical of G (see, e.g. [2] 0.8). The derived series $N = N_1 \supset N_2 = [N_1, N_1] \supset N_3 = [N_2, N_2] \supset \cdots \supset N_n = (0)$ has quotients which are commutative and unipotent, i.e., vector spaces over $\mathbf{R}$. As N is normal in G, so are all the N_i. The following proposition reduces the study of acceptable groups to the reductive case:

PROPOSITION 1.3: *G is acceptable if and only if M is.*

Proof: Suppose G is acceptable. Let ϕ_1, ϕ_2: $\Gamma \to M$ denote element-conjugate homomorphisms, and let ϕ_i' be obtained by composing ϕ_i with the injection $M \subset G$. Then

$$\phi_2' = g\phi_1' g^{-1}, g = nm.$$

As N is a normal subgroup of G,

$$\phi_2'(\gamma) = n(m\phi_1'(\gamma)m^{-1})n^{-1} = m\phi_1'(\gamma)m^{-1}n_\gamma' \quad \forall \gamma \in \Gamma.$$

As $M \cap N = \{1\}$, $n_\gamma' = 1$, ϕ_1 and ϕ_2 are conjugate in M. Suppose, conversely, that M is acceptable. Let $M_i = G/N_i$, so $M_1 = M$. We use induction on n to prove that $M_n = G$ is acceptable. The induction step requires the following claim: if M' is acceptable, and

$$1 \to V \to M'' \to M' \to 1,$$

where V is a commutative unipotent group, then M'' is acceptable. Consider a pair of element-conjugate homomorphisms ϕ_1, ϕ_2: $\Gamma \to M''$. The compositions ϕ_1' and ϕ_2' of ϕ_1 and ϕ_2 with $M'' \to M'$ are conjugate, so without loss of generality we may assume that they are equal. They define an action of Γ on V. It is easily checked that

$$c_\gamma = \phi_1(\gamma)\phi_2(\gamma)^{-1} \in V \cong \mathbf{R}^n,$$

is a 1-cocycle in $H^1(\Gamma, V)$ and a coboundary if and only if ϕ_1 and ϕ_2 are conjugate by an element of V. But $\mathbf{R}[\Gamma]$ is a semisimple algebra, so all the higher cohomology groups vanish. The proposition follows. ∎

PROPOSITION 1.4: *Let G be the group of real points of a connected reductive algebraic group, G^+ its identity component in the strong topology, $Z(G)$ its center, and D its derived group. If $Z(G)G^+ = G$, then G is acceptable if and only if D is.*

Proof: As G is reductive, $DZ(G)$ contains G^+ and hence is all of G. Thus $Z(D) = Z(G) \cap D$, so

$$D^{ad} \stackrel{\text{def}}{=} D/Z(D) \tilde{\to} G/Z(G) \stackrel{\text{def}}{=} G^{ad}$$

is an isomorphism. This implies that two elements of D are D-conjugate if and only if they are G-conjugate. Therefore, if G is acceptable, so is D. Suppose, conversely, that D is acceptable. Let ϕ_1, $\phi_2\colon \Gamma \to G$ be element-conjugate. As D is normal in G,

$$\{\gamma \in \Gamma \mid \phi_1(\gamma) \in D\} = \{\gamma \in \Gamma \mid \phi_2(\gamma) \in D\} \stackrel{\text{def}}{=} \Gamma'.$$

The restrictions $\phi_1|_{\Gamma'}$ and $\phi_2|_{\Gamma'}$ are therefore element D-conjugate and hence globally D-conjugate. Without loss of generality, therefore, the restrictions may be assumed equal. As G/D is commutative, the compositions of ϕ_1 and ϕ_2 with $G \to G/D$ must coincide, so

$$\phi_1(x)\phi_2(x)^{-1} \in D$$

for all x. Define

$$\tilde{\Gamma} = \{(\gamma, d) \in \Gamma \times D \mid \bar{\phi}_2(\gamma) = \bar{d}\},$$

where $\bar{}$ denotes projection from a group to its adjoint group, and let $\tilde{\phi}_2\colon \tilde{\Gamma} \to D$ denote the projection pr_2. Although the square in the diagram

$$\begin{array}{ccccc} \tilde{\Gamma} & \xrightarrow{\tilde{\phi}_2} & D & & \\ {\scriptstyle pr_1}\downarrow & \times & \downarrow & \searrow & \\ \Gamma & \xrightarrow{\phi_2} & G & \longrightarrow & G^{ad} = D^{ad} \end{array}$$

does not necessarily commute, the two homomorphisms from $\tilde{\Gamma}$ to D^{ad} are the same, so the obstruction to commutation,

$$\eta(x) = \tilde{\phi}_2(x)^{-1}\phi_2(pr_1(x)),$$

is a homomorphism from $\tilde{\Gamma}$ to $Z(G)$. Define

$$\tilde{\phi}_1(x) = \phi_1(pr_1(x))\eta(x)^{-1} = \phi_1(pr_1(x))\phi_2(pr_1(x))^{-1}\tilde{\phi}_2(x).$$

As $\phi_1(y)\phi_2(y)^{-1} \in D$, we have $\tilde{\phi}_1(\tilde{\Gamma}) \subset D$. By hypothesis, there exist elements $g_x \in D$ for all $x \in \Gamma$ such that

$$\phi_1(x) = g_x\phi_2(x)g_x^{-1}.$$

As $\eta(x) \in Z(G)$ for $x \in \tilde{\Gamma}$,

$$\tilde{\phi}_1(x) = \phi_1(pr_1(x))\eta(x)^{-1} = g_{pr_1(x)}\phi_2(pr_1(x))g_{pr_1(x)}^{-1}\eta(x)^{-1} = g_{pr_1(x)}\tilde{\phi}_2(x)g_{pr_1(x)}^{-1}.$$

Since D is acceptable, there exists $d \in D$ such that

$$d\tilde{\phi}_1(x)d^{-1} = \tilde{\phi}_2(x) \quad \forall x \in \tilde{\Gamma}.$$

Therefore,

$$d^{-1}\phi_2(pr_1(x))d = d^{-1}\tilde{\phi}_2(x)\eta(x)d = \tilde{\phi}_1(x)\eta(x) = \phi_1(pr_1(x))$$

for all $x \in \tilde{\Gamma}$. As $\tilde{\Gamma}$ maps onto Γ, this implies that ϕ_1 and ϕ_2 are globally G-conjugate. ∎

COROLLARY 1.5: *If G is the group of complex points of a connected reductive algebraic group, and D is its derived group, then G is acceptable if and only if D is so.*

PROPOSITION 1.6: *Let $G_{\mathbf{R}}$ denote a real algebraic group and $H_{\mathbf{R}}$ a subgroup. If $G(\mathbf{R})$ is compact, then two elements of $G(\mathbf{R})$ are $H(\mathbf{R})$-conjugate if and only if, as elements of $G(\mathbf{C})$, they are $H(\mathbf{C})$-conjugate.*

Proof: One direction is obvious. The other is proved by adapting the method of [6] 2.2.2. For any irreducible real representation ρ of G, let V_ρ denote the corresponding left G-module and W_ρ the right G-module associated to V_ρ^*. Then, as a (G, G)-bimodule, $L^2(G, \mathbf{R})$ decomposes

$$L^2(G, \mathbf{R}) = \bigoplus_\rho V_\rho \otimes W_\rho.$$

As a vector space, $V_\rho \otimes W_\rho \cong \mathrm{Hom}_{\mathbf{R}}(V_\rho, V_\rho)$; the subspace on which (h, h^{-1}) acts trivially for all $h \in H$ is $\mathrm{Hom}_H(V_\rho, V_\rho)$. Every ρ extends to a complex representation, $\tilde{\rho}$, of $G(\mathbf{C})$, and every element of $\mathrm{Hom}_{H(\mathbf{R})}(V_\rho, V_\rho)$ extends to an element of

$$\mathrm{Hom}_{H(\mathbf{C})}(V_{\tilde{\rho}}, V_{\tilde{\rho}}),$$

and thus to a holomorphic function on $G(\mathbf{C})$ invariant by $H(\mathbf{C})$-conjugation. As $H(\mathbf{R})$ is compact, $H(\mathbf{R})$-orbits are closed, so any two distinct orbits can be separated by a square integrable function on $G(\mathbf{R})$ invariant by conjugation by $H(\mathbf{R})$. Some component $\alpha \in \mathrm{Hom}_{H(\mathbf{R})}(V_\rho, V_\rho)$ must therefore seperate the orbits. As the function extends to a $H(\mathbf{C})$-invariant function on $G(\mathbf{C})$, the corresponding $H(\mathbf{C})$-orbits must be distinct. ∎

PROPOSITION 1.7: *If G is the group of complex points of a connected reductive algebraic group and K is a maximal compact subgroup of G, then K is acceptable if and only if G is acceptable.*

Proof: Suppose K is acceptable and ϕ_1, ϕ_2: $\Gamma \to G$ are element G-conjugate. We can take K to be the group of real points of an algebraic group of which G is the group of complex points. As all maximal compact subgroups of G are conjugate, the finite groups $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ can be conjugated to lie in K. By Proposition 1.6, two elements of K are K-conjugate if and only if they are G-conjugate. Hence, ϕ_1 and ϕ_2 can be taken to be element K-conjugate, hence globally K-conjugate, hence globally G-conjugate.

Conversely, suppose G is acceptable and ϕ_1, ϕ_2: $\Gamma \to K$ are element K-conjugate. Then they are element G-conjugate, hence globally G-conjugate. Let H be the real algebraic group such that $H(\mathbf{R}) = K$ and $H(\mathbf{C}) = G$. Apply Proposition 1.6 to the diagonal inclusion

$$H \subset H \times H \times \cdots \times H \cong H^{\Gamma},$$

to conclude that $\gamma \to \phi_1(\gamma)$ and $\gamma \to \phi_2(\gamma)$ are globally $K = H(\mathbf{R})$-conjugate. ∎

2. Acceptable groups

This section proves that many simple groups, including the classical groups $\mathrm{SL}(n, \mathbf{C})$, $O(n, \mathbf{C})$, and $\mathrm{Sp}(2n, \mathbf{C})$, are acceptable. Some of the results seem to be already known, but for lack of references, we work from scratch.

PROPOSITION 2.1: *The groups $\mathrm{GL}(n, \mathbf{C})$ and $\mathrm{SL}(n, \mathbf{C})$ are acceptable for all $n \geq 1$.*

Proof: If ϕ_1, ϕ_2: $\Gamma \to \mathrm{GL}(n, \mathbf{C})$ are element-conjugate, the characters associated to these two representations of Γ are the same. Therefore, the isomorphism classes of the representations are the same, which means ϕ_1 and ϕ_2 are globally $\mathrm{GL}(n, \mathbf{C})$-conjugate. The case of $\mathrm{SL}(n, \mathbf{C})$ follows from Proposition 1.4. ∎

PROPOSITION 2.2: *The groups $\mathrm{GL}(n, \mathbf{R})$ are acceptable for all n. The groups $\mathrm{SL}(n, \mathbf{R})$ are acceptable for $n = 2$ and for odd n.*

Proof: Let $G = \mathrm{GL}(n, \mathbf{R})$, and ϕ_1, ϕ_2: $\Gamma \to G$ element-conjugate homomorphisms. Decompose $\mathbf{C}^n$ as a $\phi_2(\Gamma)$-module into isotypic components $V_i =$

$W_i \otimes \mathbf{C}^{n_i}$, where the W_i are irreducible. By Schur's lemma,

$$H = Z_{\mathrm{GL}(n,\mathbf{C})}\phi_2(\Gamma) = \prod_i \mathrm{GL}(n_i, \mathbf{C}).$$

Let ρ denote the permutation such that $\bar{V}_i = V_{\rho(i)}$. Thus complex conjugation exchanges the i^{th} and $\rho(i)^{th}$ components of H when $i \neq \rho(i)$ and acts in the usual way when $i = \rho(i)$. In particular,

$$H^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})} \cong \left(\prod_{i<\rho(i)} \mathrm{GL}(n_i, \mathbf{C})\right) \times \left(\prod_{i=\rho(i)} \mathrm{GL}(n_i, \mathbf{R})\right).$$

If $\phi_{i,\mathbf{C}}$ denotes the composition of ϕ_i with $\mathrm{GL}(n, \mathbf{R}) \hookrightarrow \mathrm{GL}(n, \mathbf{C})$, then $\phi^\natural_{1,\mathbf{C}} = \phi^\natural_{2,\mathbf{C}}$, so $\phi_{1,\mathbf{C}}$ and $\phi_{2,\mathbf{C}}$ are conjugate. Choose $g \in \mathrm{GL}(n, \mathbf{C})$ such that

$$\phi_1(\gamma) = g\phi_2(\gamma)g^{-1} \quad \forall\gamma \in \Gamma.$$

Conjugating both sides, we deduce

$$\phi_2(\gamma) = g^{-1}\bar{g}\phi_2(\gamma)\bar{g}^{-1}g,$$

so $g^{-1}\bar{g} \in H$. Let a_σ: $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \to H$ denote the 1-cocycle $\sigma \mapsto g^{-1}\sigma(g)$ (in non-abelian cohomology). If it is a coboundary $h^{-1}\sigma(h)$ for some $h \in H$, then $hg^{-1} \in H^{\mathrm{Gal}(\mathbf{C}/\mathbf{R})} \subset G$, and

$$hg^{-1}\phi_1(\gamma)gh^{-1} = g^{-1}\phi_1(\gamma)g = \phi_2(\gamma),$$

so ϕ_1 and ϕ_2 are globally G-conjugate.

We conclude by showing that $H^1(\mathrm{Gal}(\mathbf{C}/\mathbf{R}), K(\mathbf{C})) = 0$, when the algebraic group K is either $\mathrm{GL}(n)$, or the Weil restriction $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathrm{GL}(n)$. The first case follows from a suitably generalized form of Hilbert's theorem 90 ([7] I 5.2). The second case is easy to check: the cocycle condition for a 1-cochain $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) \to \mathrm{GL}(n, \mathbf{C}) \times \mathrm{GL}(n, \mathbf{C})$ is $a_\sigma a_\sigma^\sigma = 1$, where σ denote complex conjugation. Equivalently, $a_\sigma = (A, A^{-1})$ for some invertible complex matrix. Then a_σ is the coboundary of $b = (A^{-1}, I)$. This proves the acceptability of $\mathrm{GL}(n, \mathbf{R})$. The acceptability of $\mathrm{SL}(n, \mathbf{R})$ for odd n follows from Proposition 1.4. For $n = 2$, the unitarian trick shows that every finite subgroup of $\mathrm{SL}(2, \mathbf{R})$ can be conjugated into a fixed subgroup $\mathrm{SO}(2, \mathbf{R})$ and is therefore cyclic. Element-conjugacy of a generator implies global conjugacy. ∎

PROPOSITION 2.3: *For all n, $O(n,\mathbf{C})$ is acceptable; if n is odd, $\mathrm{SO}(n,\mathbf{C})$ is acceptable.*

Proof: If ϕ_1, $\phi_2\colon \Gamma \to \mathrm{SO}(n,\mathbf{C})$ are element-conjugate, their compositions with the inclusion i: $\mathrm{SO}(n,\mathbf{C}) \hookrightarrow \mathrm{GL}(n,\mathbf{C})$ are globally $\mathrm{GL}(n,\mathbf{C})$-conjugate. As n-dimensional representations of Γ, $i\phi_1$ and $i\phi_2$ have the same decomposition into isotypic components

$$\bigoplus_{i=1}^{m} V_i \otimes \mathbf{C}^{m_i} \cong \bigoplus_{i=1}^{m} V_i' \otimes \mathbf{C}^{m_i}, \tag{2.3.1}$$

where the representations $V_i \cong V_i'$ are irreducible. The inclusion i gives each side of (2.3.1) a symmetric inner product. The existence of an isomorphism of Γ-modules preserving the inner product implies the $O(n,\mathbf{C})$-conjugacy of ϕ_1 and ϕ_2. By restriction, each $V_i \otimes \mathbf{C}^{m_i}$, and hence each V_i, acquires a symmetric self-duality which respects the Γ-action. Likewise, each V_i' inherits a symmetric self-duality. It suffices to show that, up to scalar multiplication, there is at most one self-duality which respects a given irreducible Γ-representation V. But by Schur's lemma,

$$1 \geq \dim(\mathrm{Hom}_\Gamma(V^*, V)) = \dim\left((V^{\otimes 2})^\Gamma\right) \geq \dim\left(\mathrm{Sym}^2(V)^\Gamma\right).$$

Therefore, $O(n,\mathbf{C})$ is acceptable. If n is odd, $SO(n,\mathbf{C})\{\pm 1\} = O(n,\mathbf{C})$, so $O(n,\mathbf{C})$-conjugacy implies $SO(n,\mathbf{C})$-conjugacy. ∎

PROPOSITION 2.4: *For all n, $\mathrm{Sp}(2n,\mathbf{C})$ is acceptable.*

Proof: If ϕ_1, $\phi_2\colon \Gamma \to \mathrm{Sp}(2n,\mathbf{C})$ are element-conjugate, their compositions with the inclusion i: $\mathrm{Sp}(2n,\mathbf{C}) \hookrightarrow \mathrm{GL}(2n,\mathbf{C})$ are globally $\mathrm{GL}(2n,\mathbf{C})$-conjugate. Decompose ϕ_1 and ϕ_2 as $2n$-dimensional representations of Γ, to obtain a Γ-module isomorphism

$$\bigoplus_{i=1}^{m} V_i \otimes \mathbf{C}^{m_i} \cong \bigoplus_{i=1}^{m} V_i' \otimes \mathbf{C}^{m_i} \tag{2.4.1}$$

Each side is endowed with a perfect anti-symmetric duality which respects the action of Γ. The restriction of this pairing to a factor $V_i \otimes \mathbf{C}^{m_i}$ is either perfect or zero. As Γ acts trivially on $\mathbf{C}^{m_i}$, the space of Γ-homomorphisms $V_i \otimes \mathbf{C}^{m_i} \to (V_i \otimes \mathbf{C}^{m_i})^*$ is a free $\mathrm{End}(\mathbf{C}^{m_i})$-module. If V is an irreducible Γ-representation,

$$\begin{aligned} 1 &\geq \dim_{\mathrm{End}(\mathbf{C}^{m_i})}(\mathrm{Hom}_\Gamma((V_i \otimes \mathbf{C}^{m_i})^*, V_i \otimes \mathbf{C}^{m_i})) \\ &\geq \dim_{\mathrm{End}(\mathbf{C}^{m_i})}\left(\bigwedge^2 (V_i \otimes \mathbf{C}^{m_i})^\Gamma\right). \end{aligned}$$

Therefore, up to the action of $\mathrm{End}(\mathbf{C}^{m_i})$, there is at most one structure of perfect anti-symmetric pairing on $V_i \otimes \mathbf{C}^{m_i}$ which respects the action of Γ. If the pairing on $V_i \otimes \mathbf{C}^{m_i}$ is zero, there must be a corresponding dual factor $V_j \otimes \mathbf{C}^{m_j}$, $m_i = m_j$ in the direct sum decomposition (2.4.1). Then any Γ-isomorphism between $V_i \otimes \mathbf{C}^{m_i}$ and $V_i' \otimes \mathbf{C}^{m_i}$ admits a unique extension to a symplectic Γ-isomorphism

$$V_i \otimes \mathbf{C}^{m_i} \oplus V_j \otimes \mathbf{C}^{m_j} \cong V_i' \otimes \mathbf{C}^{m_i} \oplus V_j' \otimes \mathbf{C}^{m_j}.$$

Therefore, ϕ_1 and ϕ_2 are globally $\mathrm{Sp}(2n, \mathbf{C})$-conjugate, by the argument of Proposition 2.3. ∎

COROLLARY 2.5: *The unitary group* $\mathrm{SU}(n)$, *and the compact orthogonal and symplectic groups,* $\mathrm{SO}(2n+1, \mathbf{R})$ *and* $\mathrm{Sp}(2n, \mathbf{C}) \cap \mathrm{SU}(2n)$, *are acceptable.*

Proof: Immediate from Propositions 2.1, 2.3 and 2.4 by virtue of Proposition 1.7. ∎

LEMMA 2.6: *Let A denote a fixed element of $U(n)$ and X_A the set*

$$X_A = \{M \in U(n) |\ A\,{}^tMA^{-1} = M\}.$$

Then every element of X_A has a square root in X_A.

Proof: Choose $M \in X_A$. As M is unitary, it is diagonalizable: $M = BDB^{-1}$, where B is unitary and D is unitary and diagonal. Choose, for each complex number which appears as a diagonal entry of D, a fixed logarithm, and use these choices to construct a diagonal matrix L such that $D = e^L$ and D and L have the same centralizer in $\mathrm{GL}(n, \mathbf{C})$. As D is unitary, L is purely imaginary. Now,

$$D = B^{-1}MB = B^{-1}A\,{}^tMA^{-1}B = B^{-1}A\,{}^tB^{-1}D\,{}^tBA^{-1}B,$$

so the matrix $B^{-1}A\,{}^tB^{-1}$ lies in the centralizer of D and therefore in the centralizer of L. In particular, it commutes with $e^{\frac{L}{2}}$. Therefore, $N = Be^{\frac{L}{2}}B^{-1}$, which is evidently a square root of M, satisfies $A\,{}^tNA^{-1} = N$. On the other hand, N is the product of the exponential of the skew symmetric matrix $\frac{L}{2}$ with unitary matrices. Hence $N \in X_A$. ∎

LEMMA 2.7: *If T denotes complex conjugation, then $G = \mathrm{SU}(3) \rtimes \langle T \rangle$ is acceptable.*

Proof: Consider element-conjugate homomorphisms $\phi_1, \phi_2\colon \Gamma \to G$. The compositions of ϕ_i with $G \to \langle T \rangle \cong \mathbf{Z}/2\mathbf{Z}$ are element-conjugate, hence identical. Let

$$\Gamma' = \phi_1^{-1}(\mathrm{SU}(3)) = \phi_2^{-1}(\mathrm{SU}(3)),$$

and let ϕ_i' denote the restriction of ϕ_i to Γ'. For each $\gamma' \in \Gamma'$, $\phi_1'(\gamma')$ is conjugate either to $\phi_2'(\gamma')$ or $\bar{\phi}_2'(\gamma')$. By Corollary 2.5 (abusing notation by writing ϕ_i' also for the associated 3-dimensional representation of Γ') $\phi_1' \oplus \bar{\phi}_1'$ and $\phi_2' \oplus \bar{\phi}_2'$ are equivalent 6-dimensional representations of Γ'. If either ϕ_i' is irreducible, this means that ϕ_1' is globally SU(3)-conjugate to ϕ_2' or $\bar{\phi}_2'$ and in either case globally G-conjugate to ϕ_2'. If not, decompose the unitary representations:

$$\phi_1' = \alpha \oplus \beta, \quad \phi_2' = \alpha \oplus \bar{\beta}.$$

By conjugating ϕ_2' if necessary by T, α may be taken to be the 2-dimensional factor. Since the 9-dimensional unitary representation $\phi_i' \otimes \bar{\phi}_i'$ doesn't depend on i,

$$\alpha \otimes \bar{\beta} \oplus \bar{\alpha} \otimes \beta \cong \alpha \otimes \beta \oplus \bar{\alpha} \otimes \bar{\beta}.$$

If α is irreducible, its tensor product with any character is irreducible, so $\alpha \otimes \bar{\beta}$ is isomorphic to $\alpha \otimes \beta$ or $\bar{\alpha} \otimes \bar{\beta}$. Thus, there are three possibilities: α is itself reducible; $\alpha = \bar{\alpha}$ (in which case ϕ_1' is globally SU(3)-conjugate to $\bar{\phi}_2'$); or $\alpha \not\cong \bar{\alpha}$ but $\alpha \otimes \beta^2 = \alpha$.

Consider the case that α is reducible, *i.e.*, that $\alpha \cong \gamma \oplus \delta$, where $\beta\gamma\delta \cong 1$. Matching characters in

$$\bar{\beta}\gamma \oplus \bar{\beta}\delta \oplus \beta\bar{\gamma} \oplus \beta\bar{\delta} \cong \beta\gamma \oplus \beta\delta \oplus \bar{\beta}\bar{\gamma} \oplus \bar{\beta}\bar{\delta},$$

either $\beta\gamma \cong \bar{\beta}\gamma$, which implies $\beta \cong \bar{\beta}$; $\beta\gamma \cong \bar{\beta}\delta$, which implies $\beta \cong \delta^2$, $\gamma \cong \delta^{-3}$; $\beta\gamma \cong \beta\bar{\gamma}$, which implies $\gamma \cong \bar{\gamma}$; or $\beta\gamma \cong \beta\bar{\delta}$, which implies $1 \cong \beta\gamma\delta \cong \beta \cong \bar{\beta}$. If β, γ, and δ are all powers of a common character, the ϕ_i' factor through a cyclic group, so element-G-conjugacy implies global-G-conjugacy without further ado. If $\beta \cong \bar{\beta}$, then $\phi_1' \cong \phi_2'$. The remaining possibility is that $\gamma \cong \bar{\gamma}$. But by symmetry, we can also conclude $\delta \cong \bar{\delta}$, and therefore $\phi_1' \cong \bar{\phi}_2'$.

If α is irreducible, it must be of dihedral, tetrahedral, octahedral, or icosahedral type, according to its composition with $U(2) \to \mathrm{PSU}(2) = \mathrm{SO}(3, \mathbf{R})$. If $\beta \cong \bar{\beta}$, we are done immediately, so we may assume $\beta^2 \not\cong 1$. Since $\alpha \otimes \beta^2 \cong \alpha$ we have $\mathrm{Tr}(\alpha(x)) = 0$ for all x outside the proper subgroup $\ker(\beta^2)$ of Γ'. This rules out all but the dihedral case, and every unitary character on a dihedral group is real. Thus $\alpha \cong \bar{\alpha}$, contrary to hypothesis.

Without loss of generality, then, the restrictions of ϕ_1 and ϕ_2 to Γ' may be taken to coincide. If $\Gamma = \Gamma'$, we are done at this point, so without loss of generality we may assume $\Gamma/\Gamma' = \mathbf{Z}/2\mathbf{Z}$. From this point on, it no longer matters that the ϕ_i are element-conjugate; it is enough that they are homomorphisms. Let

$$\epsilon(x) = \phi_1(x)\phi_2(x)^{-1} = \begin{cases} 1 & \text{if } x \in \Gamma', \\ \epsilon & \text{if } x \notin \Gamma', \end{cases}$$

where ϵ lies in the centralizer, Z, of $\phi_1(\Gamma') = \phi_2(\Gamma')$ in SU(3). We seek $z \in Z$ such that ϕ_1 and ϕ_2 are conjugate by z. The conjugation action of $\phi_i(\gamma)$ on Z depends only on the image of γ in $\Gamma/\Gamma' = \mathbf{Z}/2\mathbf{Z}$, so this is equivalent to saying that the non-abelian 1-cocycle defined by ϵ in $H^1(\mathbf{Z}/2\mathbf{Z}, Z)$ is trivial. We have the following cases:

(i) ϕ_1 is irreducible; $Z = \mu_3$.

(ii) $\phi_1 = \alpha \oplus \beta$; $Z = U(1)$.

(iii) ϕ_1 is a sum of three distinct characters; $Z = T$, a maximal torus.

(iv) ϕ_1 is a sum of two characters, one with multiplicity 2; $Z = S(U(2)\times U(1)) = U(2)$.

(v) ϕ_1 is three times a single character; $Z = G$.

To see the action of $\mathbf{Z}/2\mathbf{Z}$ on the centralizer, it is useful to conjugate so that $\phi_1(\Gamma)$ lies in a block-diagonal subgroup of SU(3). The generator of $\mathbf{Z}/2\mathbf{Z}$ acts by the product of the transpose-inverse automorphism and $ad(n)$, where n lies in the normalizer $N_{\mathrm{SU}(3)}\phi_1(\Gamma)$. In cases (i)–(iii) we have ordinary (abelian) group cohomology. In case (i), the cohomology is killed by 2 and 3 and is therefore trivial. For cases (ii) and (iii), the compact torus T is the quotient of its universal cover by its cocharacter group $X_*(T)$, so

$$H^1(\mathbf{Z}/2\mathbf{Z}, T) \cong H^2(\mathbf{Z}/2\mathbf{Z}, X_*(T)) \cong X_*(T)^{\mathbf{Z}/2\mathbf{Z}}/\mathrm{Tr}_{\mathbf{Z}/2\mathbf{Z}}X_*(T).$$

For case (ii), the cohomology vanishes because the action of $\mathbf{Z}/2\mathbf{Z}$ is non-trivial. For case (iii), the action can be any outer automorphism of the root system A_2 which is of order 2, and it is easily checked that in each case the cohomology is trivial. Cases (iv) and (v) follow from Lemma 2.6. Indeed, since the action of

$\mathbf{Z}/2\mathbf{Z}$ (whether on elements of $U(2)$ or of $\mathrm{SU}(3)$) is given by $M \mapsto A\,{}^t\!M^{-1}A^{-1}$ for some unitary matrix A, a 1-cocycle is given by an element M such that $MA\,{}^t\!M^{-1}A^{-1} = I$, i.e., such that $M \in X_A$. Therefore, there exists $N \in X_A$ with $N^2 = M$. The boundary of the 0-cycle N^{-1} is $A\,{}^t\!NA^{-1}N = N^2 = M$, so the cohomology is trivial. ∎

PROPOSITION 2.8: *The complex Lie group G_2 and its compact form are acceptable.*

Proof: In this case, the compact case is easier, and the complex case follows from Proposition 1.7. Let $G = G_2$ be compact, and let $\phi_1, \phi_2\colon \Gamma \to G$ denote element-conjugate homomorphisms. Compose the ϕ_i with the 7-dimensional orthogonal representation ρ of G_2 (the action on traceless real Cayley numbers) to obtain element conjugate homomorphisms $\rho\phi_1$ and $\rho\phi_2$ from Γ to $\mathrm{SO}(7,\mathbf{R})$. By 2.5, there exists $g \in \mathrm{SO}(7,\mathbf{R})$ such that

$$\rho(\phi_1(\gamma)) = g\rho(\phi_2(\gamma))g^{-1} \quad \forall\gamma \in \Gamma.$$

Thus,

$$\rho(\phi_1(\Gamma)) \subset \rho(G) \cap g\rho(G)g^{-1}. \tag{2.8.1}$$

This intersection of 14-dimensional subgroups of a 21-dimensional group must have dimension ≥ 7. Its identity component is therefore a compact connected Lie group of reductive rank ≤ 2 and dimension ≥ 7 which admits a homomorphism to G_2. By classification, the only groups of this kind are $U(3)$, which is the stabilizer of a non-zero vector in ρ, and $G = G_2$ itself. As G_2 has no outer automorphisms, its normalizer in $\mathrm{SO}(7,\mathbf{R})$ equals its centralizer in $\mathrm{SO}(7,\mathbf{R})$, which, as the representation is irreducible, is $G_2Z(\mathrm{SO}(7,\mathbf{R})) = G_2$. It follows that if the intersection 2.8.1 is isomorphic to G, then $g \in G$, so ϕ_1 and ϕ_2 are globally G-conjugate. It suffices to treat the case $\left(\rho(G) \cap g\rho(G)g^{-1}\right)^\circ \cong \mathrm{SU}(3)$. The identity component is always a normal subgroup, so

$$\rho(G) \cap g\rho(G)g^{-1} \subset N_G(\mathrm{SU}(3)).$$

The normalizer of $\mathrm{SU}(3)$ is $\mathrm{SU}(3) \rtimes \langle T\rangle$, in the notation of Lemma 2.7; it consists of the subgroup of G which stabilizes a *line* in the space of traceless Cayley numbers V.

Thus $\rho(\phi_1(\Gamma))$ is contained in the stabilizer $\mathrm{SU}(3)\rtimes\langle T\rangle$ of some line L_1 in V, and likewise $\rho(\phi_2(\Gamma))$ is contained in the stabilizer of some line L_2. As G acts transitively on the unit sphere in V, without loss of generality $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ lie in the same subgroup $\mathrm{SU}(3)\rtimes\langle T\rangle$ of G. If these two homomorphisms are element $\mathrm{SU}(3)\rtimes\langle T\rangle$-conjugate, the proposition follows from Lemma 2.7. In fact, by virtue of the proof of 2.7, it suffices to prove element-conjugacy on the subgroup $\Gamma' = \phi_1^{-1}(\mathrm{SU}(3)) = \phi_2^{-1}(\mathrm{SU}(3))$. Now, every element of SU(3) is conjugate to an element in its diagonal torus $T = S(U(1)\times U(1)\times U(1))$. As G is of rank 2, this is also a maximal torus of G. Elements $x, y \in T$ are G-conjugate if and only $y = w(x)$ for some $w \in W_G$. As $W_G = D_6 = \langle -1\rangle D_3 = \langle -1\rangle W_{\mathrm{SU}(3)}$, this is equivalent to the condition $y = w(x^{\pm 1})$ for x and y to be conjugate in $\mathrm{SU}(3)\rtimes\langle T\rangle$. ∎

3. Unacceptable groups

In this section we give several examples of unacceptable simple groups. Comparing the following results with those of §2, we see that acceptability is preserved neither by twisting nor by isogeny. Proposition 3.3 is due to Serre [1] but the rest seems to be new.

3.1. Let

$$0 \to Z \to \tilde{G} \to G \to 0$$

denote a central extension of real Lie groups and $\alpha \in H^2(G, Z)$ the corresponding class in group cohomology (see, *e.g.*, [4], Ch. XIV, §4.) Given a homomorphism $\phi\colon \Gamma \to G$, the cohomology class

$$\phi^*(\alpha) \in H^2(\Gamma, Z)$$

is invariant under global G-conjugation of ϕ. Indeed, lifting $\phi(\gamma)$, $\phi(\delta)$, and $\phi(\gamma\delta)$ to elements $\tilde{\gamma}$, $\tilde{\delta}$, $\widetilde{\gamma\delta}$ respectively, $\phi^*(\alpha)$ is given by the 2-cocycle

$$z_{\gamma,\delta} = \tilde{\gamma}\tilde{\delta}\widetilde{\gamma\delta}^{-1}.$$

If $\phi_2 = g\phi_1 g^{-1}$ then we can lift g to $\tilde{g}$ and lift each $g\gamma g^{-1}$ to $\tilde{g}\tilde{\gamma}\tilde{g}^{-1}$ to obtain

$$z_{2,\gamma,\delta} = \tilde{g} z_{1,\gamma,\delta}\tilde{g}^{-1} = z_{1,\gamma,\delta}.$$

We use this cohomology class in sections 3.3–3.5 and 3.7 to show that certain element-conjugate homomorphisms are not globally conjugate.

LEMMA 3.2: *Two semisimple elements g_1, $g_2 \in \mathrm{GL}(n, \mathbf{C})$ are conjugate if and only if $\mathrm{Tr}(g_1^n) = \mathrm{Tr}(g_2^n)$ for all $n \in \mathbf{N}$.*

Proof: Two semisimple elements of $\mathrm{GL}(n, \mathbf{C})$ are conjugate if and only their characteristic polynomials are the same. The Newton identities allow us to write the elementary symmetric functions as polynomials in the power sums, so the latter determine the former. ∎

PROPOSITION 3.3: *If $m \geq 1$, $n > 2$, then $\mathrm{GL}(mn, \mathbf{C})/\mu_n$ and $\mathrm{SL}(mn, \mathbf{C})/\mu_n$ are unacceptable.*

Proof: Let $\Gamma = (\mathbf{Z}/n\mathbf{Z})^2$. The Heisenberg group, $X(n)$, i.e., the group of upper triangular 3×3 unipotent matrices with entries in $\mathbf{Z}/n\mathbf{Z}$, is a central extension of Γ by $\mathbf{Z}/n\mathbf{Z}$. Let ρ_1 and ρ_2 denote irreducible (n-dimensional) complex representations of $X(n)$ whose central characters, χ_1 and χ_2 respectively, are primitive and different. Explicitly, ρ_j maps the generators of Γ to

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e(a_j/n) & 0 & \cdots & 0 \\ 0 & 0 & e(2a_j/n) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e(-a_j/n) \end{pmatrix},$$

where $e(z) = e^{2\pi i z}$, and a_j is prime to n. The matrix $\alpha^a\beta^b$ is traceless unless both a and b are divisible by n, so $\mathrm{Tr}(\rho_j(x)) = 0$ for every $x \in X(n)$ outside the center $Z(X(n)) \cong \mathbf{Z}/n\mathbf{Z}$. Let k denote the lowest positive integer such that $x^k \in Z(X(n))$. As

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & ka & k(b + (k-1)ac/2) \\ 0 & 1 & kc \\ 0 & 0 & 1 \end{pmatrix},$$

and χ_1 and χ_2 are primitive, we have

$$\chi_2(x^k)\chi_1(x^k)^{-1} = \alpha^k$$

for some $\alpha \in \mu_n$. This is completely clear if n is odd, and still true if n is even, since $\chi_2\chi_1^{-1}$ is then the square of a character of $\mathbf{Z}/n\mathbf{Z}$. Therefore,

$$\mathrm{Tr}(\rho_2(x)^i) = \mathrm{Tr}(\alpha\rho_1(x)^i)$$

for all i. It follows that the associated homomorphisms $\sigma_i\colon \Gamma \to \mathrm{GL}(n,\mathbf{C})/\mu_n$ are element-conjugate. If they were actually globally $\mathrm{GL}(n,\mathbf{C})/\mu_n$-conjugate, however, the homomorphisms ρ_i, $i = 1, 2$, would coincide up to a character of $X(n)$. Since every character of $X(n)$ is trivial on $Z(X(n)) = [X(n), X(n)]$, this is impossible. More generally, letting ϕ_i^m, $i = 1, 2$, denote the homomorphisms obtained from $\underbrace{\rho_i \oplus \cdots \oplus \rho_i}_{m \text{ times}}$ by dividing through by μ_n, the two homomorphisms are element-conjugate in $\mathrm{GL}(mn,\mathbf{C})$ (even in $\mathrm{GL}(n,\mathbf{C})^m/\mu_n$), but not globally $\mathrm{GL}(mn,\mathbf{C})/\mu_n$-conjugate. ∎

COROLLARY 3.4: *$\mathrm{PSO}(6n,\mathbf{C})$ is unacceptable.*

Proof: Immediate from $\mathrm{PSO}(6,\mathbf{C}) \cong \mathrm{PSL}(4,\mathbf{C})$. By comparison,

$$\mathrm{PSO}(4,\mathbf{C}) \cong \mathrm{SO}(3,\mathbf{C}) \times \mathrm{SO}(3,\mathbf{C})$$

is acceptable.

Propositions 2.1 and 3.3 leave open the acceptability of quotients of $\mathrm{GL}(2n,\mathbf{C})$ or $\mathrm{SL}(2n,\mathbf{C})$ by μ_2. For $n = 1$ the isomorphism $\mathrm{SL}(2,\mathbf{C})/\mu_2 \cong \mathrm{SO}(3,\mathbf{C})$ gives an affirmative answer in light of Proposition 2.3. The rest of our knowledge is contained in the following proposition. ∎

PROPOSITION 3.5: *For all $n \in \mathbf{N}$, $\mathrm{GL}(16n,\mathbf{C})/\mu_2$ is unacceptable.*

Proof: Let $\Gamma = (\mathbf{Z}/4\mathbf{Z})^2$. Consider the central extension, X, of Γ, order 32, defined by the element

$$[z_{x,y}] = [(-1)^{x\wedge y}] \in H^2(\Gamma, \mu_2). \tag{3.5.1}$$

It is easily checked that this represents a non-zero cohomology class. Choose an associated set of liftings $\tilde{x} \in X$ of elements of Γ such that the identity of Γ lifts to the identity of X. The order of $\tilde{x}$ is always the same as that of x, since $z_{x,kx} = 0$ for all $k \in \mathbf{N}$. It follows that the generator z of $\ker(X \to \Gamma)$ is not a power of any other element of X. Let ρ_1 denote the regular representation $\mathbf{C}[\Gamma]$ of Γ, viewed as an X-representation, and let ρ_2 denote $\ker(\mathbf{C}[X] \to \mathbf{C}[\Gamma])$. These are both 16-dimensional representations, and they are element-conjugate on $X \setminus \{z\}$. Indeed,

$$\mathrm{Tr}(\rho_1(x)) = \mathrm{Tr}(\rho_2(x)) = 0,$$

for all $x \notin \{1, z\}$, so by Proposition 3.2, it suffices to recall that z is not a power of any other element of X.

Now consider the maps $\phi_i\colon \Gamma \to \mathrm{GL}(16,\mathbf{C})/\mu_2$ obtained from ρ_i by passing to the quotient. They are certainly element-conjugate. By 3.1, they cannot be globally conjugate since ρ_1 lifts to a homomorphism $\Gamma \to \mathrm{GL}(16,\mathbf{C})$ and therefore defines a trivial class in $H^2(\Gamma,\mu_2)$ while ρ_2 does not lift and therefore defines a non-trivial class. This disposes of the case $n = 1$. More generally, writing ρ_i^n for $\rho_i \otimes \mathbf{C}^n$, and ϕ_i^n for the the quotient homomorphism $\Gamma \to \mathrm{GL}(16n,\mathbf{C})/\mu_2$, the same argument shows that ρ_1^n and ρ_2^n are element-conjugate but not globally conjugate. ∎

LEMMA 3.6: *For all $n \in \mathbf{N}$,*

(i) *two semisimple elements, x, $y \in \mathrm{GSp}(2n,\mathbf{C}) \subset \mathrm{GL}(2n,\mathbf{C})$ are conjugate in $\mathrm{GSp}(2n,\mathbf{C})$ if they have the same multiplier and are $\mathrm{GL}(2n,\mathbf{C})$-conjugate.*

(ii) *two semisimple elements, x, $y \in \mathrm{SO}(2n,\mathbf{C}) \subset \mathrm{GL}(2n,\mathbf{C})$ are conjugate in $\mathrm{SO}(2n,\mathbf{C})$ if they are conjugate in $\mathrm{GL}(2n,\mathbf{C})$, and, as elements of $\mathrm{GL}(2n,\mathbf{C})$, have at least one eigenvalue ± 1.*

Proof: As $x \in \mathrm{GSp}(2n,\mathbf{C})$ is semisimple, it belongs to a maximal torus, T_{GSp}, which is a subset of a maximal torus T_{GL} of $\mathrm{GL}(2n)$. We can choose coordinates on $T_{\mathrm{GL}} \cong \mathbf{C}^{*\,2n}$ so that T_{GSp} is defined by

$$x_1x_2 = x_3x_4 = \cdots = x_{2n-1}x_{2n}.$$

Every semisimple $y \in \mathrm{GSp}(2n)$ (resp. $\mathrm{GL}(2n)$) is $\mathrm{GSp}(2n)$-conjugate (resp. $\mathrm{GL}(2n)$-conjugate) to an element of T_{GSp} (resp. T_{GL}), so to prove (i), it suffices to show that two elements of T_{GSp} are GSp-conjugate if and only if they are GL-conjugate. Two elements of T_{GSp} are GSp- (resp. GL-) conjugate if and only if they lie in the same orbit of the Weyl group $W_{\mathrm{GSp}} \cong (\mathbf{Z}/2\mathbf{Z})^n \rtimes S_n$ (resp. $W_{\mathrm{GL}} \cong S_{2n}$). In particular, two $2n$-tuples $(x_1,\ldots,x_{2n})$, $(y_1,\ldots,y_{2n}) \in T_{\mathrm{GSp}}$ are W_{GSp}-conjugate if and only if they have the same unordered set of unordered pairs of coordinates:

(3.6.1)
$$\{\{x_1,x_2\},\{x_3,x_4\},\ldots,\{x_{2n-1},x_{2n}\}\} = \{\{y_1,y_2\},\{y_3,y_4\},\ldots,\{y_{2n-1},y_{2n}\}\}.$$

Any permutation of the x_i which preserves the values x_1x_2, x_3x_4, $\cdots$, $x_{2n-1}x_{2n}$ preserves the set of pairs (3.6.1). This gives (i).

For $\mathrm{SO}(2n,\mathbf{C})$, $T_{\mathrm{SO}} \subset T_{\mathrm{GL}}$ is defined by

$$x_1x_2 = x_3x_4 = \cdots = x_{2n-1}x_{2n} = 1.$$

Two elements of T_{SO} are conjugate if and only if they lie in the same orbit of the Weyl group $W_{\mathrm{SO}} \cong H \rtimes S_n$, where $H \subset (\mathbf{Z}/2\mathbf{Z})^n$ is the subgroup of n-tuples whose entries sum to zero. Therefore, two sets of ordered pairs

$$\{(x_1, x_1^{-1}), (x_3, x_3^{-1}), \ldots, (x_{2n-1}, x_{2n-1}^{-1})\}$$

correspond to conjugate elements of SO$(2n)$ if and only if one can be obtained from the other by replacing an even number of ordered pairs (x, x^{-1}) by (x^{-1}, x) If some $x_i = x_i^{-1} = \pm 1$, this condition is equivalent to (3.6.1). This gives (ii). ∎

PROPOSITION 3.7: *For all* $n \in \mathbf{N}$, PSp$(16n, \mathbf{C})$ *and* PSO$(16n, \mathbf{C})$ *are unacceptable.*

Proof: Recall the groups Γ and X and the representations ρ_i^n defined in Proposition 3.5. It suffices to show that $\mathbf{C}^{16n}$ admits an orthogonal (resp. symplectic) inner product which ρ_1 and ρ_2 respect (resp. up to scalar multiplication), and that $\rho_1(x)$ is conjugate to $\rho_2(x)$ in SO$(16n, \mathbf{C})$ (resp. GSp$(16n, \mathbf{C})$.) It is enough to check this when $n = 1$. A regular representation is always orthogonal (taking $\{[x] |\ x \in X\}$ as an orthonormal basis of $\mathbf{C}[X]$), so the ρ_i are orthogonal. Since every cyclic subgroup is of even index in X, the ρ_i land in SO$(16, \mathbf{C})$. If $x_0 \in X$ is a central element of order 2, and $\epsilon\colon X \to U(1)$ is a character such that $\epsilon(x_0) = -1$, then up to scalar multiplication, X respects the symplectic pairing

$$\langle [x], [y] \rangle = \epsilon(x) \delta_{x_0 x, y}$$

on $\mathbf{C}[X]$. If $x_0 \notin \ker(X \to \Gamma)$, we may choose ϵ to factor through $X \to \Gamma$. Thus, the restriction of $\langle\ ,\ \rangle$ to the representation spaces of ρ_1 and ρ_2 are perfect. Therefore, both ρ_1 and ρ_2 can be taken to land in GSp$(16, \mathbf{C})$, and both have multiplier ϵ.

By the proof of Proposition 3.5, $\rho_1^n(x)$ and $\rho_2^n(x)$ are conjugate for all $x \notin \ker(X \to \Gamma)$, so by Lemma 3.6 (i), they are likewise element GSp-conjugate. Every element of X has order dividing 4, and the square of an element of order 4 is not in $\ker(X \to \Gamma)$ (and hence has at least one eigenvalue of 1). By 3.6 (ii), therefore, outside $\ker(X \to \Gamma)$, the ρ_i^n are element SO-conjugate. Therefore the quotients $\phi_i^n\colon \Gamma \to \mathrm{GL}(16n, \mathbf{C})/\mu_2$ of the ρ_i^n are element GSp$/\mu_2$-conjugate (resp. element PSO-conjugate) homomorphisms. They cannot be globally conjugate because they are not even globally GL$(16n, \mathbf{C})/\mu_2$-conjugate. We conclude that PSO$(16n, \mathbf{C})$ and GSp$(16n, \mathbf{C})/\mu_2$ are unacceptable. By Corollary 1.5, PSp$(16n, \mathbf{C})$ is unacceptable as well. ∎

PROPOSITION 3.8: *For all even $n \geq 8$, $SL(n,\mathbf{R})$, $\mathrm{SO}(n,\mathbf{R})$, and $\mathrm{SO}(n,\mathbf{C})$ are unacceptable.*

Proof: Let σ denote the Steinberg representation of $\mathrm{SL}(3,\mathbf{Z}/2\mathbf{Z})$ (see χ_6 in [5] p. 3). The Frobenius-Schur indicator is 1, so σ can be viewed as a homomorphism to $O(8,\mathbf{R})$. The value of $\mathrm{Tr}(\sigma)$ on elements of order 2, 3, 4, and 7 is 0, -1, 0, and 1 respectively. From this, we deduce that the eigenvalues of $\sigma(g)$ are given by the table

Order of g	Eigenvalues of $\sigma(g)$
1	1, 1, 1, 1, 1, 1, 1, 1
2	1, 1, 1, 1, -1, -1, -1, -1
3	1, 1, ω, ω, ω, ω^2, ω^2, ω^2
4	1, 1, i, i, -1, -1, $-i$, $-i$
7	1, 1, ζ, ζ^2, ζ^3, ζ^4, ζ^5, ζ^6

where ω and ζ denote $e^{2\pi i/3}$ and $e^{2\pi i/7}$ respectively. In particular, the image of σ lies in $\mathrm{SO}(8,\mathbf{R})$.

Let τ denote an injective homomorphism $\mathbf{Z}/4\mathbf{Z} \to \mathrm{SO}(2,\mathbf{R})$. The centralizer of $\tau(\mathbf{Z}/4\mathbf{Z})$ in $\mathrm{GL}(2,\mathbf{R})$ is the group

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2+b^2>0 \right\}.$$

Let τ^k denote the $2k$-dimensional orthogonal representation of $(\mathbf{Z}/4\mathbf{Z})^k$ in which the i^{th} $\mathbf{Z}/4\mathbf{Z}$-factor acts on the i^{th} $\mathbf{R}^2$ summand. We set

$$\Gamma = \mathrm{SL}(3,\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/4\mathbf{Z})^{n/2-4}; \quad \phi_1 = \sigma \oplus \tau^{n/2-4}\colon \Gamma \to \mathrm{SO}(n,\mathbf{R}).$$

Let M denote any orthogonal $n \times n$ matrix with determinant -1, and let

$$\phi_2 = M\phi_1 M^{-1}\colon \Gamma \to \mathrm{SO}(n,\mathbf{R}).$$

If $\phi_2 = N\phi_1 N^{-1}$ for some $N \in \mathrm{SL}(n,\mathbf{R})$, then $M^{-1}N$ is a real $n\times n$ matrix with negative determinant commuting with $\phi_1(\Gamma)$. All real matrices which commute with $\tau(\mathbf{Z}/4\mathbf{Z})$ have positive determinant, and likewise all matrices which commute with $\sigma(\mathrm{SL}(3,\mathbf{Z}/2\mathbf{Z}))$. Therefore all real matrices commuting with $\phi_1(\Gamma)$ have positive determinant, so ϕ_1 and ϕ_2 are not globally $\mathrm{SL}(n,\mathbf{R})$-conjugate. *A fortiori*, they are not globally $\mathrm{SO}(n,\mathbf{R})$-conjugate. On the other hand, for any

$\alpha \in \Gamma$, $\phi_1(\alpha)$ fixes some vector v_α. Let N_α denote the reflection through $v_\alpha^\perp$. Then $N_\alpha M^{-1} \in \mathrm{SO}(n, \mathbf{R})$, and

$$\phi_2(\alpha) = M\phi_1(\alpha)M^{-1} = MN_\alpha^{-1}\phi_1(\alpha)N_\alpha M^{-1}$$

is $\mathrm{SO}(n, \mathbf{R})$-conjugate to $\phi_1(\alpha)$. We conclude that $\mathrm{SO}(n, \mathbf{R})$ and $\mathrm{SL}(n, \mathbf{R})$ are unacceptable. By Proposition 1.7, $\mathrm{SO}(n, \mathbf{C})$ is unacceptable. ∎

LEMMA 3.9: *Let π denote the quotient map $\mathrm{Spin}(2n+1, \mathbf{C}) \to \mathrm{SO}(2n+1, \mathbf{C})$ and ρ: $\mathrm{SO}(2n+1, \mathbf{C}) \to \mathrm{GL}(2n+1, \mathbf{C})$ the standard representation. If x is a semisimple element of $\mathrm{SO}(2n+1, \mathbf{C})$ such that -1 is an eigenvalue of $\rho(x)$, then the two elements of $\pi^{-1}(x)$ are conjugate to one another in $\mathrm{Spin}(2n+1, \mathbf{C})$.*

Proof: Let T_{SO} denote a maximal torus of $\mathrm{SO}(2n+1, \mathbf{C})$ which contains x. Every commutator in $\pi^{-1}(T_{\mathrm{SO}})$ lies in $\pi^{-1}(1)$, so the identity component of $\pi^{-1}(T_{\mathrm{SO}})$ is an n-dimensional commutative group of semisimple elements of $\mathrm{Spin}(2n+1, \mathbf{C})$. Therefore $\pi^{-1}(T_{\mathrm{SO}})^\circ$ is a maximal torus of $\mathrm{Spin}(2n+1, \mathbf{C})$. Every maximal torus of a connected semisimple group contains the center, so

$$\pi^{-1}(T_{\mathrm{SO}})^\circ = \pi^{-1}(T_{\mathrm{SO}}).$$

We call this group T_{Spin}.

Let $\tilde{T}$ denote the universal covering group of a complex torus T. Then $T \cong \mathbf{C}^{\mathrm{rk}(T)}$. In particular,

$$\tilde{T}_{\mathrm{SO}} = \tilde{T}_{\mathrm{Spin}} = \mathbf{C}^n.$$

As $\pi_1(T) = X_*(T)$, the fibration $\mathbf{Z}/2\mathbf{Z} \to T_{\mathrm{Spin}} \to T_{\mathrm{SO}}$ gives rise to the homotopy sequence

$$0 \to X_*(T_{\mathrm{Spin}}) \to X_*(T_{\mathrm{SO}}) \to \mathbf{Z}/2\mathbf{Z} \to 0$$

and its dual

$$0 \to X^*(T_{\mathrm{SO}}) \to X^*(T_{\mathrm{Spin}}) \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

Now $\mathrm{SO}(2n+1, \mathbf{C})$ and $\mathrm{Spin}(2n+1, \mathbf{C})$ are the adjoint and simply connected Lie groups respectively with Lie algebra $\mathfrak{so}(2n+1, \mathbf{C})$. In standard coordinates, therefore,

$$X^*(T_{\mathrm{SO}}) = \mathbf{Z}\epsilon_1 + \cdots + \mathbf{Z}\epsilon_n, \quad X^*(T_{\mathrm{Spin}}) = \left\{ \sum_{i=1}^n a_i\epsilon_i \;\middle|\; a_i - a_j \in \mathbf{Z},\ 2a_i \in \mathbf{Z}\ \forall i, j \right\}$$

([3] Planche II). In the dual basis ϵ_i^* of ϵ_i,

$$X_*(T_{\mathrm{SO}}) = \mathbf{Z}\epsilon_1^* + \cdots + \mathbf{Z}\epsilon_n^*, \quad X_*(T_{\mathrm{Spin}}) = \left\{ \sum_{i=1}^n a_i\epsilon_i^* \;\middle|\; \sum_{i=1}^n a_i \in 2\mathbf{Z} \right\}.$$

The quotient map

$$q\colon \mathbf{C}^n \to \mathbf{C}^n/X_*(T_{\mathrm{SO}}) = T_{\mathrm{SO}} = (\mathbf{C}^*)^n$$

sends

$$(z_1, \dots, z_n) \mapsto \left(e^{2\pi i z_1}, \dots, e^{2\pi i z_n}\right).$$

The eigenvalues of $\rho(q(z_1, \dots, z_n))$ are therefore 1, $e^{\pm 2\pi i z_1}, \dots, e^{\pm 2\pi i z_n}$.

The Weyl group of $\mathrm{Spin}(2n+1, \mathbf{C})$ with respect to T_{Spin} is $W = \{\pm 1\}^n \rtimes S_n$. The action of W on T_{Spin} is most simply described at the level of the covering space $\tilde{T}_{\mathrm{Spin}} = \mathbf{C}^n$: $\sigma \in S_n$ acts by permuting coordinates and $(w_1, \dots, w_n) \in (\mathbf{Z}/2\mathbf{Z})^n$ acts by element-wise multiplication. Choose

$$(z_1, \dots, z_n) \in q^{-1}(x) \subset \tilde{T}_{\mathrm{SO}} = \tilde{T}_{\mathrm{Spin}}.$$

By hypothesis, some eigenvalue $e^{\pm 2\pi i z_j} = -1$, so some $z_j \in \frac{1}{2} + \mathbf{Z}$. Let $w \in W$ denote the simple reflection through the j^{th} coordinate hyperplane. Then

$$w\left(\sum_{i=1}^n z_i\epsilon_i^*\right) = \left(\sum_{i \neq j} z_i\epsilon_i^*\right) - z_j\epsilon_j^* \equiv \left(\sum_{i=1}^n z_i\epsilon_i^*\right) - \epsilon_j^* \pmod{X_*(T_{\mathrm{Spin}})}.$$

Therefore, w fixes x but permutes the two elements of $\pi^{-1}(x)$. Choosing a representative n of w in the normalizer N of T_{Spin} in $\mathrm{Spin}(2n+1, \mathbf{C})$, we conclude that the two elements of $\pi^{-1}(x)$ are conjugate by n in $\mathrm{Spin}(2n+1, \mathbf{C})$. ∎

PROPOSITION 3.10: *For all $n \geq 9$, $\mathrm{Spin}(n, \mathbf{C})$ is unacceptable.*

Proof: Let M_{10} denote the (non-simple) Mathieu group in A_{10}, and let ρ denote the restriction of the standard 9-dimensional representation of A_{10} to M_{10}. For information about M_{10} and $\mathrm{Tr}(\rho)$ we consult [5] p. 5, where they are denoted $A_6.2_3$ and χ_6 respectively; see also *loc. cit.* pp. 48–49, for information about the connection with A_{10}. The Frobenius-Schur indicator is 1, so ρ is orthogonal. The values of $\mathrm{Tr}(\rho)$ on elements of order 2, 3, 4, 5, 8 are 1, 0, 1, −1, −1 respectively, so the eigenvalues of $\rho(g)$ are given by the following table:

Order of g	Eigenvalues of $\rho(g)$
1	$1, 1, 1, 1, 1, 1, 1, 1, 1$
2	$1, 1, 1, 1, 1, -1, -1, -1, -1$
3	$1, 1, 1, \omega, \omega, \omega, \omega^2, \omega^2, \omega^2$
4	$1, 1, 1, i, i, -1, -1, -i, -i$
5	$1, \zeta, \zeta, \zeta^2, \zeta^2, \zeta^3, \zeta^3, \zeta^4, \zeta^4$
8	$1, \xi, i, \xi^3, -1, -1, \xi^5, -i, \xi^7$

where ω, ζ, and ξ denote $e^{2\pi i/3}$, $e^{2\pi i/5}$, and $e^{\pi i/4}$ respectively. In particular, $\rho(\Gamma)$ lies in $\mathrm{SO}(9, \mathbf{R})$. We define Γ and ϕ_1 to make the following square cartesian:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\phi_1} & \mathrm{Spin}(9, \mathbf{C}) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ M_{10} & \xrightarrow{\rho} & \mathrm{SO}(9, \mathbf{C}). \end{array}$$

Let $\epsilon\colon \Gamma \to \mathrm{Spin}(9, \mathbf{C})$ denote the composite map

$$\Gamma \xrightarrow{\pi} M_{10} \twoheadrightarrow M_{10}/A_6 = \mathbf{Z}/2\mathbf{Z} = \ker(\mathrm{Spin}(9, \mathbf{C}) \to \mathrm{SO}(9, \mathbf{C})) \hookrightarrow \mathrm{Spin}(9, \mathbf{C}),$$

and set $\phi_2(x) = \phi_1(x)\epsilon(x)$.

Suppose, for some $g \in \mathrm{Spin}(9, \mathbf{C})$, $\phi_2 = g\phi_1 g^{-1}$. Let ψ denote the composition

$$\mathrm{Spin}(9, \mathbf{C}) \to \mathrm{SO}(9, \mathbf{C}) \to \mathrm{GL}(9, \mathbf{C}).$$

Then

$$\psi(g)\psi(\phi_1(\gamma))\psi(g)^{-1} = \psi(\phi_2(\gamma)) = \psi(\phi_1(\gamma)) \quad \forall \gamma \in \Gamma$$

so

$$\psi(g)\rho(m)\psi(g)^{-1} = \rho(m) \quad \forall m \in M_{10}.$$

As ρ is irreducible, this means $\psi(g)$ is a scalar. The only scalar orthogonal matrix in $\mathrm{SO}(9, \mathbf{C})$ is the identity, so g must lie in the center of $\mathrm{Spin}(9, \mathbf{C})$. This is impossible, since ϵ is non-trivial. Thus ϕ_1 and ϕ_2 are not globally conjugate. On the other hand, $\phi_1(\gamma) = \phi_2(\gamma)$ when $\pi(\gamma)$ is the square of an element of M_{10}, in particular, if $\pi(\gamma)$ is of odd order. On the other hand, by Lemma 3.9 and the table of eigenvalues above, $\phi_1(\gamma)$ is conjugate to $\phi_2(\gamma)$ if $\pi(\gamma)$ is of even order. Thus $\mathrm{Spin}(9, \mathbf{C})$ is unacceptable.

For all $n \geq 10$, we realize $\mathrm{SO}(9, \mathbf{C})$ as the subgroup of $\mathrm{SO}(n, \mathbf{C})$ which stabilizes pointwise a subspace $H \subset \mathbf{C}^n$ of codimension 9. There is a corresponding covering homomorphism $\mathrm{Spin}(9,\mathbf{C}) \to \mathrm{Spin}(n,\mathbf{C})$. To see that it does not factor through $\mathrm{SO}(9,\mathbf{C})$, it suffices to note that the restriction of the spin representation from $\mathrm{Spin}(2k+1,\mathbf{C})$ to $\mathrm{Spin}(2k)$ is the sum of the semispin representations, while each semispin representation of $\mathrm{Spin}(2k,\mathbf{C})$ restricts to the spin representation on $\mathrm{Spin}(2k-1,\mathbf{C})$. If $g \in \mathrm{SO}(n,\mathbf{C})$, $g\,\mathrm{SO}(9,\mathbf{C})g^{-1}$ is the stabilizer of gH, so

$$\mathrm{SO}(9,\mathbf{C}) \cap g\,\mathrm{SO}(9,\mathbf{C})g^{-1} = \mathrm{Stab}_{\mathrm{SO}(n,\mathbf{C})}(H+gH).$$

This intersection is of the form $\mathrm{SO}(m,\mathbf{C})$, where $m \le 9$, and $\mathrm{SO}(m,\mathbf{C})$ is the stabilizer of $(gH+H)/H$ in $\mathrm{SO}(9,\mathbf{C})$. Moreover $m=9$ if and only if $gH=H$, in which case g acts on $\mathrm{SO}(9,\mathbf{C})$ by inner automorphism. It follows that at the level of covering spaces,

$$\mathrm{Spin}(9,\mathbf{C}) \cap \tilde{g}\,\mathrm{Spin}(9,\mathbf{C})\tilde{g}^{-1} \cong \mathrm{Spin}(m,\mathbf{C}),$$

where $m \le 9$ with equality if and only if $\tilde{g}$ normalizes $\mathrm{Spin}(9,\mathbf{C})$. Now, the composition of each ϕ_i with the 9-dimensional representation ψ of $\mathrm{Spin}(9,\mathbf{C})$ is irreducible. Let ϕ_i^n denote the composition of ϕ_i with the embedding $\mathrm{Spin}(9,\mathbf{C}) \hookrightarrow \mathrm{Spin}(n,\mathbf{C})$. Evidently, the homomorphisms ϕ_i^n are element-conjugate. If there exists $g \in \mathrm{Spin}(n,\mathbf{C})$ such that

$$\phi_2^n(\gamma) = g\phi_1^n(\gamma)g^{-1} \quad \forall\gamma \in \Gamma,$$

then

$$\phi_2^n(\Gamma) \subset \mathrm{Spin}(9,\mathbf{C}) \cap g\,\mathrm{Spin}(9,\mathbf{C})g^{-1} = \mathrm{Spin}(m,\mathbf{C}).$$

As ϕ_2 is irreducible, $m=9$, and g acts by an automorphism of $\mathrm{Spin}(9,\mathbf{C})$. As all automorphisms of $\mathrm{Spin}(9,\mathbf{C})$ are inner, this contradicts the fact that ϕ_1 and ϕ_2 are not conjugate. Thus $\mathrm{Spin}(n,\mathbf{C})$ is unacceptable for $n \ge 9$.

Note, by contrast, that for $n \le 6$, $\mathrm{Spin}(n,\mathbf{C})$ is acceptable, by Propositions 2.1, 1.1, and 2.4. ∎

PROPOSITION 3.11: *The complex group $F_4(\mathbf{C})$ and its compact form are unacceptable.*

Proof: By [8], one semisimple complex Lie algebra $\mathfrak{g}$ contains another such algebra $\mathfrak{h}$ of the same rank if and only if the ordinary Dynkin diagram of $\mathfrak{h}$ is obtained from the extended Dynkin diagram of $\mathfrak{g}$ by deleting a vertex. We recall ([3] Planche VIII) that the extended Dynkin diagram of F_4 looks like this:

Therefore, $\mathfrak{so}(9, \mathbf{C})$ is a Lie subalgebra of the complex Lie algebra of type F_4. The restriction of the 26-dimensional representation V_{ϖ_4} of the complex Lie algebra of type F_4 (following the notation of [3] Planche VIII) to $\mathfrak{so}(9, \mathbf{C})$ is the direct sum of the trivial representation, the standard 9-dimensional representation ψ and the spin representation. Therefore, the homomorphism $\mathrm{Spin}(9, \mathbf{C}) \to F_4(\mathbf{C})$ is injective. The image of the compact form $\mathrm{Spin}(9, \mathbf{R})$ of the spin group lies in a maximal compact subgroup $F_4(\mathbf{R})$ of $F_4(\mathbf{C})$.

We define Γ and $\phi_i \colon \Gamma \to \mathrm{Spin}(9, \mathbf{C})$ as in Proposition 3.10. By the unitarian trick, we can take ϕ_i to land in the compact real form $\mathrm{Spin}(9, \mathbf{R})$. By Proposition 1.6, ϕ_1 and ϕ_2 are element-conjugate as homomorphisms to $\mathrm{Spin}(9, \mathbf{R})$. Let ϕ_i^F denote the composition of ϕ_i with the homomorphism $\mathrm{Spin}(9, \mathbf{R}) \to F_4(\mathbf{R})$. As ϕ_1 and ϕ_2 are element-conjugate, the same is true of ϕ_1^F and ϕ_2^F. Now suppose that there exists $g \in F_4(\mathbf{R})$ such that

$$g\phi_1^F(\gamma)g^{-1} = \phi_2^F(\gamma) \ \forall \gamma \in \Gamma.$$

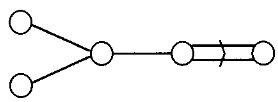
Then

$$\phi_2^F(\Gamma) \subset \mathrm{Spin}(9, \mathbf{R}) \cap g\mathrm{Spin}(9, \mathbf{R})g^{-1} = K.$$

Since all automorphisms of $\mathrm{Spin}(9, \mathbf{R})$ are inner, if g lies in the normalizer of $\mathrm{Spin}(9, \mathbf{R})$, it must decompose $g = g'z$, where z lies in the centralizer and g' lies in $\mathrm{Spin}(9, \mathbf{R})$ itself. This is impossible, since ϕ_1 and ϕ_2 are not globally $\mathrm{Spin}(9, \mathbf{R})$-conjugate. Thus K is a proper subgroup of $\mathrm{Spin}(9, \mathbf{R})$. As $\dim(\mathrm{Spin}(9, \mathbf{R})) = 36$ and $\dim(F_4(\mathbf{R})) = 52$, $\dim(K) \geq 20$. As $\phi_2^F(\Gamma)$ lies in $K \subset \mathrm{Spin}(9, \mathbf{R})$, ψ is an irreducible representation of K. Now K° is a compact, connected, linear group, and its center can be no larger than a maximal torus of $F_4(\mathbf{R})$, that is, no more than 4-dimensional. Therefore, the derived group D of K° is a compact semisimple group of dimension ≥ 16. As D is normal in K, the restriction of ψ to D is a direct sum of isotypical representations of equal dimensions and therefore either irreducible, the direct sum of three 3-dimensional representations, or the direct sum of nine 1-dimensional representations. As ψ is faithful modulo center, the same is true for $\psi|_D$. This rules out the last possibility, and the second possibility is ruled out by the dimension of D. Therefore, the complexified Lie algebra $\mathfrak{D}$ of D is a complex semisimple algebra of rank ≤ 4 with a faithful irreducible 9-dimensional representation.

Every irreducible representation of a product $\mathfrak{g} \times \mathfrak{h}$ is the exterior tensor product of irreducible representations of $\mathfrak{g}$ and $\mathfrak{h}$. Since a semisimple Lie algebra with a

3-dimensional faithful representation has dimension ≤ 8, $\mathfrak{D} = (3, \mathbf{C}) \times (3, \mathbf{C})$, or $\mathfrak{D}$ is simple. The only simple complex Lie algebras of rank ≤ 4 and dimension ≥ 16 are $(5, \mathbf{C})$, $\mathfrak{so}(7, \mathbf{C})$, $\mathfrak{so}(8, \mathbf{C})$, $\mathfrak{sp}(6, \mathbf{C})$, and $\mathfrak{sp}(8, \mathbf{C})$. None of these has an irreducible 9-dimensional representation. Finally, $(3, \mathbf{C}) \times (3, \mathbf{C})$ is not a Lie subalgebra of $\mathfrak{so}(9, \mathbf{C})$, because the extended Dynkin diagram



of $\mathfrak{so}(9, \mathbf{C})$ does not contain two disjoint simple edges. ∎

THEOREM 3.12: *The following compact groups are unacceptable:* $\mathrm{SU}(mn)/\mu_n$, $n \geq 3$; $\mathrm{SU}(16n)/\mu_2$; $\mathrm{SO}(2n, \mathbf{R})$, $n \geq 4$; $\mathrm{PSO}(6n, \mathbf{R})$; $\mathrm{PSO}(16n, \mathbf{R})$; *and the compact real forms of* $\mathrm{PSp}(16n)$ *and* $\mathrm{Spin}(n)$, $n \geq 9$.

Proof: Immediate from Propositions 3.3, 3.4, 3.5, 3.7, 3.8, and 3.10 by applying the contrapositive form of Proposition 1.7. ∎

References

[1] D. Blasius, *On multiplicities for* SL(n), Israel Journal of Mathematics, this issue, pp. 237–251.

[2] A. Borel and J. Tits, *Groupes réductifs,* Publications Mathématiques de l'IHES **27** (1965), 55–150.

[3] N. Bourbaki, *Groupes et algèbres de Lie,* Masson, Paris, 1981.

[4] H. Cartan and S. Eilenberg, *Homological Algebra,* Princeton Math Series **19**, Princeton, 1956.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups,* Clarendon Press, Oxford, 1985.

[6] P. Deligne, *La Conjecture de Weil, II,* Publications Mathématiques de l'IHES **52** (1980), 138–252.

[7] P. Deligne, *Cohomologie étale: les points de départ,* in *Cohomologie Étale*, Lecture Notes in Math. 569, Springer-Verlag, Berlin, 1977.

[8] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras,* American Mathematical Society Translations Ser. 2 **6** (1957), 111–244.

[9] T. Sunada, *Riemannian coverings and isospectral manifolds,* Annals of Mathematics **121** (1985), 169–186.